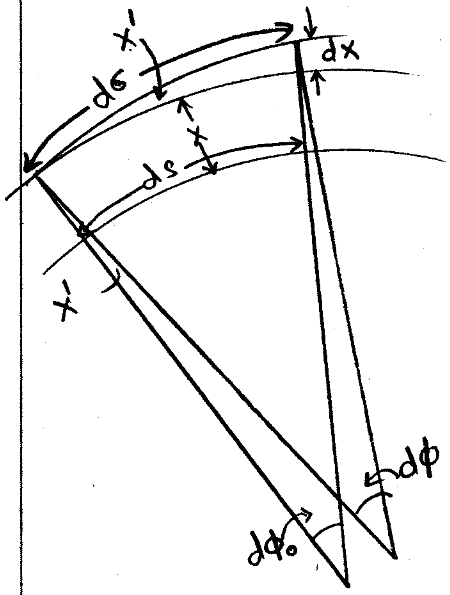
Equation of Motion in charged particle Beam Dynamics

After having defined the reference trajectory we are merely interested in the description of the deviation of an individual particle trajectory from the ideal trajectory.

In deriving the equation of motion we limit ourselves to the horizontal plane only. Let ds and $d\sigma$ represent segment of ideal particle orbit and that of individual particle trajectory as shown in the figure in a magnetic field. Then the length of the segment $d\sigma$ is given by



$$d\sigma^2 = [d\phi_0(\rho_0 + x)]^2 + dx^2$$

$$\simeq ds^2\left(1 + \frac{x}{\rho_0}\right)^2 \quad \because dx^2 \simeq 0 \;\; \& \;\; ds = d\phi_0 \cdot$$

$$\therefore \; d\sigma \simeq ds\left(1 + \frac{x}{\rho_0}\right)$$

Now $dx' = d\phi_0 - d\phi$

$$= \frac{ds}{\rho_0} - \frac{d\sigma}{\rho}$$

Substituting for $d\sigma$ we get

$$dx' = \frac{ds}{\rho_0} - \frac{ds}{\rho}\left(1 + \frac{x}{\rho_0}\right)$$

or $$\boxed{\frac{dx'}{ds} = x'' = \frac{1}{\rho_0} - \frac{1}{\rho}\left(1 + \frac{x}{\rho_0}\right)} \quad \text{— (36)}$$

Now from Eq (34) the component of magnetic field perpendicular to the plane of reference orbit is B_y and is given by

$$B_y = B_{y_0} + gX + \frac{1}{2} s(X^2 - Y^2) + \cdots$$

or the curvature is given by

$$\frac{1}{\rho} = \frac{eB}{cp} = \frac{e}{cp}\left[B_{y_0} + gX + \frac{1}{2} s(X^2 - Y^2) + \cdots\right]$$

$$= \frac{1}{\rho_0} + kX + \frac{1}{2} m(X^2 - Y^2) + \cdots \qquad (37)$$

substituting Eq^n (37) in Eq^n (36) we get

$$X'' = \frac{1}{\rho_0} - \left(1 + \frac{X}{\rho_0}\right)\left\{\frac{1}{\rho_0} + kX + \frac{1}{2} m(X^2 - Y^2) + \cdots\right\}$$

$$= -kX - \frac{1}{2} m(X^2 - Y^2) - \cdots$$

$$- \frac{X}{\rho_0^2} - \frac{kX^2}{\rho_0} - \frac{1}{2} Xm(X^2 - Y^2) - \cdots$$

Now neglecting the higher order terms like X^2, Y^2 etc., we get,

$$X'' = -\left(k + \frac{1}{\rho_0^2}\right) X$$

or $$\boxed{X'' = -KX \quad \text{where } K = k + \frac{1}{\rho_0^2}} \qquad (38)$$

similarly for vertical orbits the equation of motion will be

$$\boxed{y'' = ky} \quad \text{——} \quad (39)$$

The Equations (38) and (39) are good for monochromatic beam. But in general the particles will be having a momentum spread $\delta = \frac{\Delta p}{p}$, i.e.,

$$\frac{1}{cp} = \frac{1}{cp_o(1+\delta)} = \frac{1}{cp_o}(1-\delta+\delta^2 - \cdots) \quad \text{—} (40)$$

where $cp_o\delta$ represents a small deviation from reference momentum. Thus in horizontal plane the eq^n (37) takes the following form,

$$\frac{1}{\rho} = \frac{eB}{cp} = \frac{e}{cp_o}\left[B_{y_0} + gx + \frac{s}{2}(x^2-y^2)+ \cdots\right]\left[1-\delta+\delta^2+\cdots\right]$$

$$\frac{1}{\rho} = \frac{1}{\rho_o} - \frac{\delta}{\rho_o} + \frac{kx}{\rho_o} - \cdots \quad \text{—} (41)$$

From (36) and (41) we get

$$x'' \simeq -kx + \frac{\delta}{\rho_o} \quad \text{— by neglecting higher ordered terms}$$

Thus

$$\boxed{\begin{aligned} x'' + kx &= \frac{\delta}{\rho_o} \quad \text{— for Horizontal plane} \\ \& \quad y'' - ky &= \frac{\delta}{\rho_{oy}} \quad \text{— for Vertical plane} \end{aligned}} \quad (42)$$

These equations of motion of charged particle are of Hill's equation type. (In particular they are inhomogenious Hill's equations).

The equations in 42 can be put together as

$$u'' + K(s)u = \frac{1}{\rho_0}\frac{\Delta p}{p} \qquad \text{— } 43$$

where $K(s)$ has two terms : $\rho(s)$, bending strength $k(s)$, focusing strength. Note that this quantity $K(s)$ varies along s. This is the equation of motion for "strong focusing beam transport system", where the magnitude of the focusing strength is a free parameter.

Further, notice that the L.H.S of this equation resembles that of a harmonic oscillator with a time dependent frequency. Thus these equations describe an oscillatory motion with variable restoring force.

Solution to the Equation of Motion with $\frac{\Delta p}{p} = 0$

In our attempt to solve the equation of motion (43) we first try to solve the homogenious differential equation.

$$u'' + K(s)\,u = 0 \qquad (44)$$

The principal solution of this equation are

1. $K > 0$: $C(s) = \cos(\sqrt{K}s)$ & $S(s) = \frac{1}{\sqrt{K}}\sin(\sqrt{K}s)$ — (45a)

2. $K < 0$: $C(s) = \cosh(\sqrt{|K|}s)$ & $S(s) = \frac{\sinh(\sqrt{|K|}s)}{\sqrt{|K|}}$ — (45b)

These solutions satisfy the following initial conditions

$$\left.\begin{aligned} C(0) &= 1 & C'(0) &= \frac{dC}{ds} = 0 \\ S(0) &= 0 & S'(0) &= \frac{dS}{ds} = 1 \end{aligned}\right\} \qquad (46)$$

C and S are called cosine like and sine like functions, respectively.

Any arbitrary solution $u(s)$ can be expressed as a linear combination of these two principal solutions

$$\begin{aligned} u(s) &= C(s)\,u_0 + S(s)\,u_0' \\ u'(s) &= C'(s)\,u_0 + S'(s)\,u_0' \end{aligned} \qquad (47)$$

where u_0 and u_0' are arbitrary initial parameters of particle trajectory

In matrix formulation Eq (47) can be expressed as

$$\begin{bmatrix} u(s) \\ u'(s) \end{bmatrix} = \begin{bmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{bmatrix} \begin{bmatrix} u_0 \\ u_0' \end{bmatrix} \qquad (48)$$

The determinant of the matrix $\mathcal{D} = cs' - sc'$

By differentiating $\mathcal{D}$ with respect to s we can show that

$$(cs' - sc')' = cs'' - sc'' = 0$$

Hence the quantity $\mathcal{D}$ is independent of s.

Further

$$\begin{pmatrix} C(0) & S(0) \\ C'(0) & S'(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \text{unit matrix} \qquad (49)$$

owing to the chosen initial conditions it stays unity through out the system.

For the transformation matrix of an arbitrary beam transport line with negligible dissipating forces we get

$$\mathcal{D}(s) = \begin{vmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{vmatrix} = 1 \qquad (50)$$

Transformation Matrices of Accelerator Components

The matrix notation of the solution of the equations of motion is particularaly useful if $K(s)$ = piece wise constant. Then the solution for the complete "lattice" of optical elements is then just product of the individual matrices in the desired sequence.

There are only three cases to be considered:

a) $K > 0$ & $s = l$ (focusing)

$$\begin{bmatrix} u \\ u' \end{bmatrix}_{out} = \begin{bmatrix} \cos\sqrt{K}l & \frac{1}{\sqrt{K}}\sin\sqrt{K}l \\ -\sqrt{K}\sin\sqrt{K}l & \cos\sqrt{K}l \end{bmatrix} \begin{bmatrix} u \\ u' \end{bmatrix}_{in}$$

For pure quadrupoles $K = k$, $\frac{1}{\rho_0} = 0$

b) $K < 0$ & $s = l$ (defocusing)

$$\begin{bmatrix} u \\ u' \end{bmatrix}_{out} = \begin{bmatrix} \cosh\sqrt{K}l & \frac{1}{\sqrt{K}}\sinh\sqrt{K}l \\ \sqrt{K}\sinh\sqrt{K}l & \cosh\sqrt{K}l \end{bmatrix} \begin{bmatrix} u \\ u' \end{bmatrix}_{in}$$

c) $K = 0$ & $s = L$ (Drift space)

$$\begin{bmatrix} u \\ u' \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$$

Thin Lens Approximation :-

In many cases it is sufficient to assume the length of the magnet 'l' is small compared with focal length i.e.

$$l \ll f = \frac{1}{kl}.$$

and we set

$$l \to 0$$

keeping

$$f^{-1} = kl. = \text{constant}$$

as a consequence of that $\sqrt{k}\,l \to 0$ and the transformation matrix becomes.

$$\begin{bmatrix} u \\ u' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} u \\ u' \end{bmatrix}_{\text{initial}}$$

where $f^{-1} = kl > 0$ for focusing plane

$= kl < 0$ for defocusing plane.

Quadrupole Dublets :- This is composed of two quadrupole magnets seperated by a drift space of length L. The transformation matrix will be

$$M_{db} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_1} & 0 \end{bmatrix} = \begin{bmatrix} 1 - L/f_1 & L \\ -1/f^* & 1 - L/f_2 \end{bmatrix}$$

with $\frac{1}{f^*} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{L}{f_1 f_2}$ can be used for focusing in both planes

Lattice:-

Using the matrices above, the motion of a particle can be followed through an arrangement of accelerator elements. If a particle traverses a series of elements having matrices $M_1, M_2, \ldots M_n$ then the input and finial conditions are related by

$$M = M_n \cdots M_2 M_1$$

Lattice :- Detailed description of the way in which magnets and intervining spaces are placed to form the accelerator or a beam transport line is called "Lattice"

Stability Criterion :-

In a synchrotron or a long beam transport composed of alternatively focusing and defocusing lenses, it is not obvious at the outset what relationships between lens strength and spacing lead to stable oscillation that grow in amplitude with time

For oscillations to be stable the quantity

$$M^m \begin{pmatrix} u \\ u' \end{pmatrix}_{\text{initial}} \quad \text{must remain finite.}$$

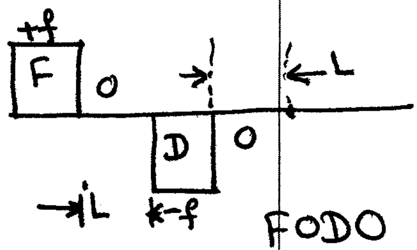
for any large value of "m".

It can be shown that the stability condition is

$$\boxed{-1 \leq \frac{\text{Trace } M}{2} \leq 1}$$

FODO Lattice :-

As an example, consider a lattice which consists only of equally spaced focusing and defocusing lenses, each of which we will assume to be thin. Such a lattice is referred to as FODO Lattice. If f is focal length of the lenses & L is drift space. Then the transformation matrix will be



$$M = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1/f & 1 \end{bmatrix}\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{L}{f} - \left(\frac{L}{f}\right)^2 & 2L + \frac{L^2}{f} \\ -\frac{L}{f^2} & 1 + \frac{L}{f} \end{bmatrix} \quad ; \quad \text{Tr}\, M = 2 - \left(\frac{L}{f}\right)^2$$

Application of stability condition gives

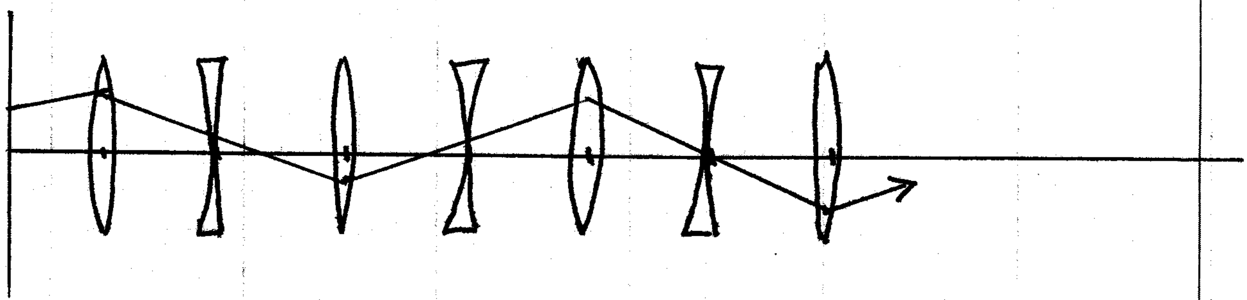
$$-1 \leq 1 - \frac{1}{2}\left(\frac{L}{f}\right)^2 \leq 1$$

or $$0 \leq 2 - \frac{1}{2}\left(\frac{L}{f}\right)^2 \leq 2$$

or $$4 \geq \left(\frac{L}{f}\right)^2$$

or $$\boxed{f \geq \frac{L}{2}}$$ $\Rightarrow$ The focal length is greater than half the lens spacing

A FODO lattice described above is a very important design element of long beam transport lines and/or accelerators. Now we can sketch an oscillation of a particle traversing a sequence of focusing and defocusing lenses



General Solution of Trajectory Equation in terms of Principal Trajectories

So far we have dealt with homogenious differential equations of Hill's type. But in reality we need to deal with inhomogenious equation explained earlier.

$$x'' + K(s)\,x = \frac{1}{\rho_0}\frac{\Delta p}{p} \qquad \text{—①}$$

The general solution of this equation can be written as

$$x(s) = C(s)\,x_0 + S(s)\,x_0' + D(s)\,\frac{\Delta p}{p_0}$$

$$x'(s) = C'(s)\,x_0 + S'(s)\,x_0' + D'(s)\,\frac{\Delta p}{p_0}$$

$C(s)$, $S(s)$ and $D(s)$ are called principal trajectories (cosine like, sinelike and dispersion).

In matrix notation

$$\begin{bmatrix} u \\ u' \end{bmatrix} = \begin{bmatrix} c & s \\ c' & s' \end{bmatrix} \begin{bmatrix} u \\ u' \end{bmatrix}_{init=0} + \frac{\Delta p}{p_0}\begin{bmatrix} D \\ D' \end{bmatrix} \qquad \text{—②}$$

or

$$\begin{bmatrix} u \\ u' \\ \frac{\Delta p}{p_0} \end{bmatrix}_s = \begin{bmatrix} c & s & D \\ c' & s' & D' \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ u' \\ \frac{\Delta p}{p_0} \end{bmatrix}_0$$

The dispersion $D(s)$ may be expressed in terms of $c(s)$ and $s(s)$ as

$$\boxed{D(s) = s\int_0^s \frac{1}{\rho}\,c\,d\tau - c\int_0^s \frac{1}{\rho}\,s\,d\tau} \qquad \text{—③}$$

The physical meaning of $D(s)\frac{\Delta p}{p_0}$ is it determines the off-set of the trajectory from the ideal path of the particles with relative energy deviation $\frac{\Delta p}{p_0}$

$$D'(s) = s' \int_0^s \frac{1}{\rho} C \, d\tau - c' \int_0^s \frac{1}{\rho} S \, d\tau \qquad \text{—— (4)}$$

The trajectory of a particle through an arbitrary beam transport system can be determined by repeated multiplication of transformation matrices explained earlier. This method is most convenient in designing modern accelerators / beam line using computers. But it does not reveal characteristic properties of particle trajectories. For deeper insite we have to solve the equation analytically.

Trajectory in terms of Amplitude and Phase Functions

The equation of motion

$$u'' + K(s)\,u = 0 \qquad \text{—— (1)}$$

has the property that the "spring constant" K is a function of the independent variable s and for an important class of accelerators K is periodic such that

$$K(s + \mathcal{C}) = K(s) \qquad \text{—— (2)}$$

The repeat distance, $\mathcal{C}$, may be as large as circumference of the accelerator or less.

The general solution to the equation of motion can be written as

$$u(s) = A\sqrt{\beta(s)}\,\cos(\phi(s) - \phi_0) \qquad \text{—— (3)}$$

$$\left.\begin{array}{l} \sqrt{\beta(s)} = \text{Amplitude} \\ \phi(s) = \text{phase} \end{array}\right\} \text{of "betatron oscillation"}$$

and they vary as a function of s nonlinearly

A is called as "Admittance" of the trajectory and is constant. $\beta(s)$ is called as "betatron function".

The differential equation for function $\beta(s)$ is

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + \beta^2 K = 1 \qquad \text{—— (4)}$$

With out getting deep into the method of derivation following expressions can be obtained.

$$\phi(s) = \int_0^s \frac{ds^*}{\beta(s^*)} + \phi_0 \qquad \text{where } s \text{ is path length.} \qquad (5)$$

This implies that the knowledge of $\beta(s)$ along the beam line obviously allows us to compute the phase function.

and

$$A = \gamma u^2 + 2\alpha u u' + \beta u'^2 \qquad (6)$$

$$\text{with } \left.\begin{aligned} \alpha &= -\frac{1}{2}\beta' \\ \gamma &= \frac{1+\alpha^2}{\beta} \end{aligned}\right\} \qquad (7)$$

The quantities α, β and γ - are called "Courant-Snyder" parameters. The eqn (6) is called "Courant-Snyder Invariant" which is an equation of ellips in u and u'. The admittance "A" is the phase space area associated with the largest ellipse that the accelerator will accept.

At any point in accelerator the maximum value of "u" is $A\sqrt{\beta}$

Phase space area occupied by the beam is called emittance

The transformation matrix then looks like

$$\begin{bmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{bmatrix} = \begin{bmatrix} \sqrt{\beta/\beta_0}\,(\cos\phi - \alpha_0 \sin\phi) & \beta\beta_0 \sin\phi \\ \dfrac{\alpha_0 - \alpha}{\sqrt{\beta\beta_0}} \cos\phi - \dfrac{1+\alpha\alpha_0}{\sqrt{\beta\beta_0}} \sin\phi & \sqrt{\beta/\beta_0}\,(\cos\phi - \alpha \sin\phi) \end{bmatrix} \quad — (7)$$

Knowledge of betatron functions along a beam line allows us to calculate individual particle trajectories. $\beta(s)$ can be obtained by solving eqn (4) or by using matrix formalism.

b) <u>Drift space</u>

$\frac{1}{\rho} = 0, k = 0$

$$M_x = M_z = \begin{pmatrix} 1 & \ell & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.14}$$

c) <u>Quadrupole Magnet</u>

The dispersion (3.13) vanishes since $\frac{1}{\rho} = 0$. With $\varphi = \ell\sqrt{|k|}$ the transformation matrices are for $k > 0$

$$\begin{aligned} M_x &= \begin{pmatrix} \cosh\varphi & \frac{1}{\sqrt{|k|}}\sinh\varphi & 0 \\ \sqrt{|k|}\sinh\varphi & \cosh\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ M_z &= \begin{pmatrix} \cos\varphi & \frac{1}{\sqrt{|k|}}\sin\varphi & 0 \\ -\sqrt{|k|}\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{3.15}$$

These matrices describe horizontal defocusing, vertical focusing.

For $k < 0$, the matrices M_x and M_z are interchanged and we get horizontal focusing, vertical defocusing.

d) <u>Thin-lens approximation</u>

In many practical cases, the focal length f of the quadrupole magnet will be much larger than the length of the lens:

$$f = \frac{1}{k\,l} \gg l$$

Then the transfer matrices can be approximated by

$$M_x = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.16}$$

$$M_z = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.17}$$

Note that these matrices describe a lens of zero length, i.e. they are derived from Eqs. (3.14) using $\ell \to 0$ while keeping $k \cdot \ell = const.$ The true length l of the lens has to be recovered by two drift spaces $\ell/2$ on either side, e.g.

$$M_z = \begin{pmatrix} 1 & \frac{\ell}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{\ell}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{\ell}{2f} & \ell-\frac{\ell^2}{4f} & 0 \\ -\frac{1}{f} & 1-\frac{\ell}{2f} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.18}$$

One might ask why the approximation has not been made by expanding $\sin\varphi$, $\cos\varphi$, etc. in Taylor series and neglecting higher powers of φ. However terminating the Taylor series at some power results in a transfer matrix whose determinant is not unity. For instance, in third order we obtain

$$M_z = \begin{pmatrix} 1-\frac{\ell}{2f} & \ell-\frac{\ell^2}{6f} & 0 \\ -\frac{1}{f} & 1-\frac{\ell}{2f} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which does not fulfil $\det M = 1$. It will be shown later that this would violate Liouville's Theorem of phase-space conservation.

For accelerators in the TeV range, where $1/\rho^2 \ll |k| \ll 1/\ell^2$, the thin-lens approximation is excellent for the matrix description of the whole accelerator.

e) Dipole sector magnet

The matrices (3.12), (3.13) with $k = 0$ describe a "hard edge" dipole, i.e. the magnet ends are perpendicular to the circular trajectory (Fig. 19).

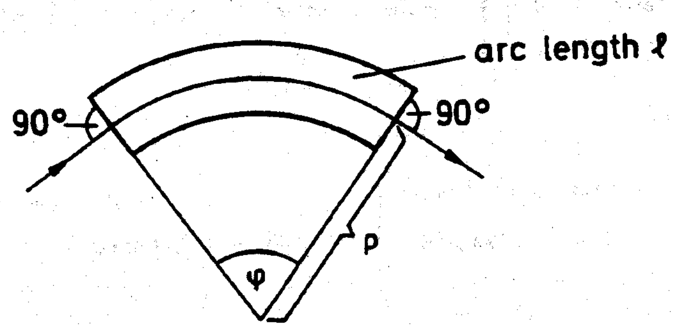


Figure 19: Dipole sector magnet

The transformation matrices are with $\varphi = \frac{\ell}{\rho}$

$$M_x = \begin{pmatrix} \cos\varphi & \rho\sin\varphi & \rho(1-\cos\varphi) \\ -\frac{1}{\rho}\sin\varphi & \cos\varphi & \sin\varphi \\ 0 & 0 & 1 \end{pmatrix} \quad M_z = \begin{pmatrix} 1 & \ell & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.19}$$

f) Rectangular dipole magnet

In practice, dipole magnets are often built straight with the magnet end plates not perpendicular to the central trajectory. A rectangular magnet can be derived from a sector magnet

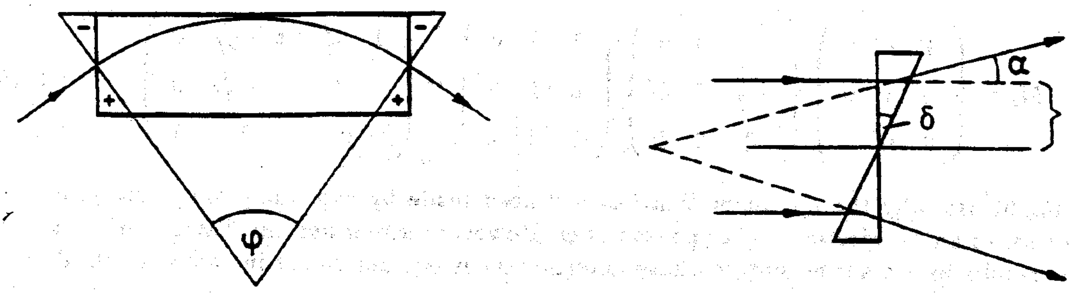


Figure 20: Rectangular dipole magnet and horizontally defocusing magnetic wedge

by superimposing at the entrance and exit a "magnetic wedge" of angle $\delta = \varphi/2$, as shown in Fig. 20.

The deflection angle in the magnetic wedge is

$$\alpha = \frac{\Delta\ell}{\rho} = \frac{x\tan\delta}{\rho} = \frac{x}{f}$$

It acts as a thin defocusing lens with $1/f = (\tan\delta)/\rho$ in the horizontal plane, as a focusing length with the same strength in the vertical plane. The horizontal transformation matrix for a rectangular magnet is

$$M_x = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\rho}\tan\delta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\varphi & \rho\sin\varphi & \rho(1-\cos\varphi) \\ -\frac{1}{\rho}\sin\varphi & \cos\varphi & \sin\varphi \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\rho}\tan\delta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For $\varphi \ll 1,\ \delta = \varphi/2:$

$$M_x = \begin{pmatrix} 1 & \rho\sin\varphi & \rho(1-\cos\varphi) \\ 0 & 1 & 2\tan\varphi/2 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_z = \begin{pmatrix} \cos\varphi & \rho\sin\varphi & 0 \\ -\frac{1}{\rho}\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.20}$$

Note that M_x is exact for $\delta = \varphi/2$ while $\varphi \ll 1$ has been used for M_z only. We conclude that in a rectangular magnet the weak horizontal focusing of a sector magnet is exactly compensated by the defocusing at the entrance and exit face. The magnet acquires, however, a weak vertical focusing of the same strength.

g) Quadrupole doublet

The transformation matrix of a system of dipoles, quadrupoles and drift spaces is obtained by multiplying the matrices of each element in the correct order. An important example is a quadrupole doublet consisting of a focusing quadrupole, a drift space and a defocusing quadrupole. Figure 21 shows two trajectories (1,2) suggesting a tendency of both horizontal and vertical focusing in this kind of arrangement.

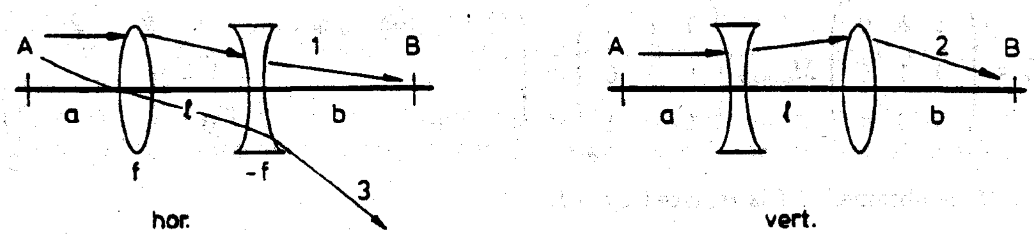


Figure 21: A quadrupole doublet consisting of a horizontally and a vertically focusing quadrupole magnet. Trajectories 1 and 2 suggest that there is a tendency of simultaneous focusing in both the horizontal and vertical directions.

The focusing action arises because trajectories entering parallel to the axis have a larger amplitude in the focusing than in the defocusing lens. Quadrupole doublets are indeed the simplest means of high energy beam focusing and imaging. We shall now derive the conditions for *simultaneous imaging in both* horizontal and vertical planes, treating the quadrupoles in the thin-lens approximation and assuming $f_{foc} = -f_{defoc} = f$ for simplicity. The horizontal transfer matrix of the doublet is (for meaning of symbols see Fig. 21)

$$M_{doub,x} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \ell & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{\ell}{f} & \ell & 0 \\ -\frac{\ell}{f^2} & 1+\frac{\ell}{f} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.21}$$

The vertical transfer matrix is obtained if f is replaced by $-f$:

$$M_{doub,x} = \begin{pmatrix} 1+\frac{\ell}{f} & \ell & 0 \\ -\frac{\ell}{f^2} & 1-\frac{\ell}{f} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.22}$$

The matrix element $M_{21} = C' = -\ell/f^2$ is called the overall refractive power of the system and it is seen to be focusing in both planes. Somewhat sloppily one could say that a beam coming from infinity (i.e. all particles perfectly parallel to the s-axis, $x_0' = 0$) will be focused in both planes, as indicated by trajectories 1 and 2 in Fig. 21. The effective focal length f_{doub} for these particles is

$$f_{doub} = \frac{f^2}{\ell} \tag{3.23}$$

Trajectory 3 in Fig. 21, however, illustrates that there are trajectories as well which are not at all bent towards the beam axis. For practical applications one might therefore ask: What happens to particles emerging from a point A at a finite distance a from the first lens? Optical imaging requires that there is a point B at a distance b behind the second lens where all particles emerging from A will converge. The horizontal transfer matrix from A to B is

$$M_x = \begin{pmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} M_{doub,x} \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{\ell}{f} - \frac{\ell b}{f^2} & a + b + \ell + \frac{\ell b}{f} - \frac{\ell a}{f} - \frac{\ell a b}{f^2} & 0 \\ -\frac{\ell}{f^2} & 1 + \frac{\ell}{f} - \frac{\ell a}{f^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.24}$$

Again, M_z is obtained if f is replaced by $-f$:

$$M_z = \begin{pmatrix} 1 + \frac{\ell}{f} - \frac{\ell b}{f^2} & a + b + \ell - \frac{\ell b}{f} + \frac{\ell a}{f} - \frac{\ell a b}{f^2} & 0 \\ -\frac{\ell}{f^2} & 1 - \frac{\ell}{f} - \frac{\ell a}{f^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.25}$$

Imaging from A to B requires $M_{12} = S \equiv 0$. Because of $\det M = 1$, the matrix can then be written in the form

$$M = \begin{pmatrix} m & 0 & 0 \\ C' & \frac{1}{m} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.26}$$

m is called magnification of object A to image B. Obviously

$$x_B = m \cdot x_A \qquad \text{(irrespective of } x'_A \text{ !)} \tag{3.27}$$

Remark: If also $M_{21} = C' = 0$ (i.e. zero overall refractive power), the system is called a telescopic system.

The condition $S \equiv 0$ is satisfied if

$$\frac{b}{f} = \frac{\frac{a}{f} + \frac{\ell}{f} - \frac{\ell a}{f^2}}{\frac{\ell a}{f^2} - \frac{\ell}{f} - 1} \qquad \text{for } M_x \tag{3.28}$$

and

$$\frac{a}{f} = \frac{\frac{b}{f} + \frac{\ell}{f} - \frac{\ell b}{f^2}}{\frac{\ell b}{f^2} - \frac{\ell}{f} - 1} \qquad \text{for } M_z \tag{3.29}$$

There are two ways to interprete Eqs. (3.27-3.29):

- If the parameter set f, ℓ, a, b is a solution in the horizontal plane, then the set f, ℓ, b, a is a solution in the vertical plane, i.e. the rôles of a and b are interchanged. This means that in general horizontal and vertical images are in different planes. This is called "astigmatic" focusing.

  Example: Consider $\ell/f = 1$ and $a/f = 3$. Then $b/f = 1$ and $m_x = -1$ for imaging in the horizontal plane while $b/f = 7/3$ and $m_z = -1/3$ for imaging in the vertical plane. The vertical solution "conjugate" to the horizontal one would be $\ell/f = 1$, $a/f = 1$, $b/f = 3$, and $m_z = -1$. The latter one means, of course, that not only the horizontal and vertical images are in different distances from the doublet but also the respective objects.

- To get the horizontal and vertical images into the *same* plane ("stigmatic" focusing), $a = b$ is required. Then

$$m_x = \frac{f + a}{f - a} = \frac{1}{m_z} \qquad \text{stigmatic focusing} \tag{3.30}$$

Equal horizontal and vertical magnifications $m_x = m_z$ are obviously impossible with stigmatic focusing, using quadrupole doublets. Note that, while $f_{foc} = -f_{defoc}$ has been assumed for Eqs. (3.21-3.30), this latter statement applies for all kinds of quadrupole doublets.

Example: $\ell/f = 4/3$, $a/f = b/f = 2$ yields stigmatic focusing with $m_x = -3$ and $m_z = -1/3$.

Figure 22 illustrates particle trajectories in a stigmatic focusing quadrupole doublet.

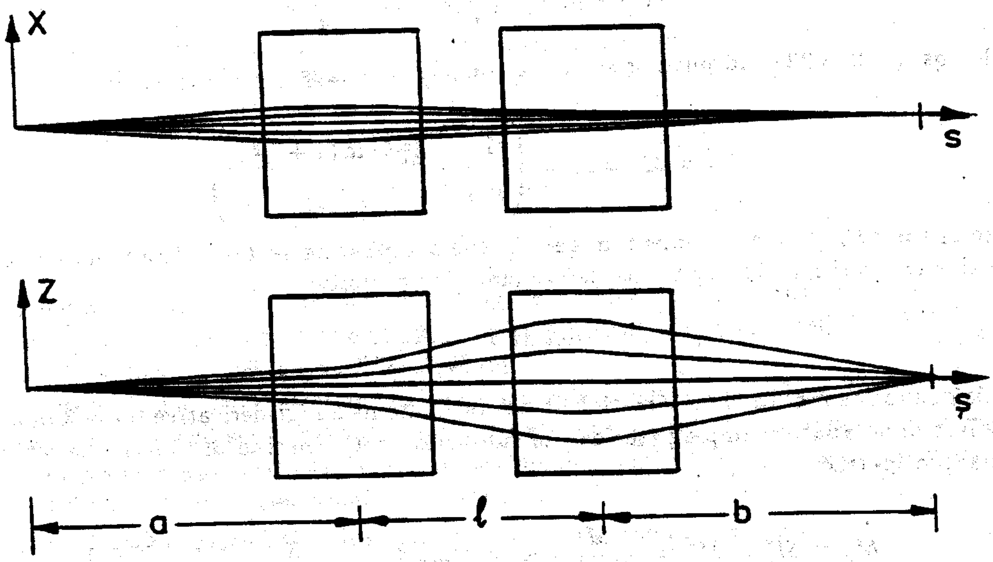


Figure 22: A stigmatic focusing quadrupole doublet showing particle trajectories that all start at the center of the object ($x_A = z_A = 0$).

In circular accelerators and for beam transport along a transfer channel we are, in general, not interested in imaging at all. Instead, we require small (or at least finite) beam envelopes for any kind of particle source, i.e. for any location of the object plane. It will be shown in section 4.7 that, to achieve focusing in an alternating series of F and D quadrupoles, the separation between two quadrupoles must not be larger than twice the focal length

$$\ell < 2|f|$$

h) Accelerating section

Acceleration in the longitudinal direction is beyond the scope of this article. However, there is also an effect on the transverse motion, which is now briefly addressed.

Consider a section of length ℓ with constant electric field E_s in the longitudinal direction. The total momentum p of an ultrarelativistic particle ($v \approx c = const$) then changes according to

$$\frac{dp}{ds} = \frac{e}{c}E_s = const \Rightarrow p(s) = p_o + \frac{e}{c}E_s \cdot s$$

$p_o = p(s = 0)$ is the momentum at the entrance of the section. The transverse motion of an ultrarelativistic particle is described by ($y = x$ or z)

$$\frac{d}{ds}p_y = \frac{d}{ds}\left(\frac{p(s)}{c}v_y\right) = \frac{d}{ds}\left(p(s)\frac{dy}{ds}\right) = 0 \tag{3.31}$$

which is a different type of differential equation than (3.5), because $p(s)$ is not constant any more. A first integral yields

$$p(s)\frac{dy}{ds} = const = y'_\circ p_\circ, \quad \text{thus} \quad \frac{dy}{ds} = y'(s) = \frac{y'_\circ p_\circ}{p_\circ + \frac{e}{c}E_s s} \tag{3.32}$$

Integrating once more we obtain

$$y(s) = y_\circ + y'_\circ \cdot \frac{cp_\circ}{eE_s} \ln\left(1 + \frac{eE_s}{cp_\circ}s\right) \tag{3.33}$$

With Eqs. (3.32,3.33) and putting $s = l$ the transfer matrix is

$$M_x = M_z = M = \begin{pmatrix} 1 & \frac{p_\circ}{\Delta p}\ell \cdot \ln\left(1 + \frac{\Delta p}{p_\circ}\right) \\ 0 & \frac{p_\circ}{p_\circ + \Delta p} \end{pmatrix} \tag{3.34}$$

where $\Delta p = \frac{e}{c}E_s \cdot l$ is the momentum gain in the accelerating section. The main complication with this transfer matrix is, that its determinant is not unity:

$$\det M = \frac{p_\circ}{p_\circ + \Delta p} \tag{3.35}$$

This is due to the fact that the equation of motion contains a first-derivative term, and it reflects the effect of adiabatic damping (see end of subsection 4.4). Instead of (3.34), one often uses a normalized matrix

$$M_x^* = M_z^* = M^* = \frac{M}{\sqrt{\det M}} = \frac{1}{\sqrt{\frac{p_\circ}{p_\circ + \Delta p}}} \begin{pmatrix} 1 & \frac{p_\circ}{\Delta p}\ell \cdot \ln\left(1 + \frac{\Delta p}{p_\circ}\right) \\ 0 & \frac{p_\circ}{p_\circ + \Delta p} \end{pmatrix}$$

which has $\det M^* = 1$. This trick saves much of the formalism to be derived in the following sections, but if it is used one has to keep in mind, that in this case the *normalized* emittance is a conserved quantity and not the usual emittance, see Eq. (4.34). A similar complication arises if dispersion is included in the 3×3 matrix version of (3.34).