Trajectory In terms of Amplitude and Phase

The equation of motion

$$u'' + K(s)\,u = 0 \qquad \text{—(1)}$$

has the property that the "spring constant" K is a function of the independent variable 's' and for important class of accelerators K is periodic such that

$$K(s + \mathcal{C}) = K(s) \qquad \text{—(2)}$$

The repeat distance $\mathcal{C}$, may be as large as circumference of the accelerator or less. For the Tevatron it is 1/6 - of the ring, MI — it is half, AC — 1/3 etc.

The solution of eq^n (1) can be written in quasi harmonic form as,

$$u(s) = A\sqrt{\beta(s)}\,\cos(\phi(s) + \phi_0) \qquad \text{—(3)}$$

$$\left.\begin{aligned} \sqrt{\beta(s)} &= \text{Amplitude} \\ \phi(s) &= \text{phase} \end{aligned}\right\} \text{ of betatron oscillations.}$$

u(s) vary as a fun of 's' nonlinearly. A is constant

Now substituting (3) in (1) we get

$$u'' + Ku = A\left\{-\tfrac{1}{2}\beta^{1/2}\beta'\phi' - \beta^{1/2}\phi'' - \tfrac{1}{2}\beta'/\beta^{1/2}\,\phi'\right\}\sin(\phi - \phi_0)$$
$$+ A\left\{-\beta^{1/2}\phi' - \tfrac{1}{4}\beta'^2/\beta^{3/2} + \tfrac{1}{2}\beta''/\beta^{1/2} + \beta^{1/2}K\right\}\cos(\phi - \phi_0) = 0$$

—(4)

This implies coefficients of sin and cos terms are independently zero. With sin - component

$$\beta \phi'' + \beta' \phi' = 0 = (\beta \phi)'$$

or $\beta \phi' =$ constant. setting constant = 1

we get

$$\phi(s) = \int_0^s \frac{ds}{\beta(s)} \quad \text{—(5)}$$

$\beta(s)$ can be treated as local (wave length/2π).

Using eq^n (5) in (4) for cos term we get

$$2\beta\beta'' - \beta'^2 + 4\beta^2 K = 4 \quad \text{—(6)}$$

<u>Courant - Snyder Invariant</u>:-

Using eq^n (3) and derivative of (3) we can eliminate cos and sin of $\phi(s)$ and ϕ_0. with the help of

$$\sin^2(\phi(s) - \phi_0) + \cos^2(\phi(s) - \phi_0) = 1 \quad \text{—(7)}$$

Then we get

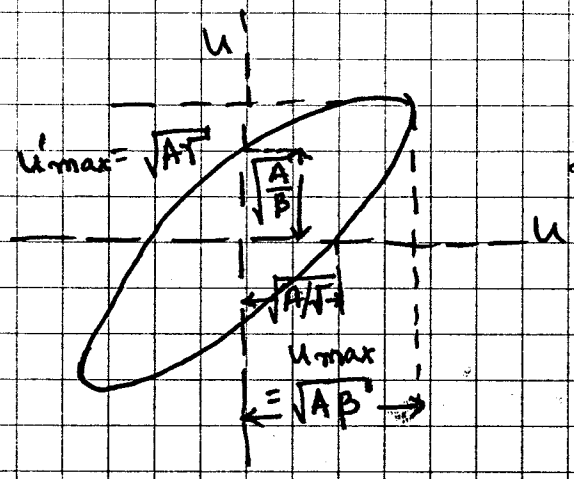
$$\gamma u^2 + 2\alpha u u' + \beta u'^2 = A^2 \quad \text{with} \quad \alpha = -\tfrac{1}{2}\beta', \ \gamma = \frac{1+\alpha^2}{\beta} \quad \text{—(8)}$$

is invariant quantity at any point in the lattice.

This is an equation of ellipse in (u, u')

For the case of circular accelerator, each time that the particle passes a perticular point in the ring

its betatron oscillations coordinates will appear as a point on the ellipse given by the amplitude funt and its slope at that point.



Notice that for different locations through the lattice, the ellipses will have different shapes and orientations but they will have same A.

<u>Admittance</u> :- This is the phase space area associated with the largest ellipse that the accelerator can accept.

<u>Transformation matrix in terms of betatron functions:</u>

Using eq^n (3) any particle trajectory can be

$$u(s) = \tilde{A}\sqrt{\beta}\cos\phi + \tilde{B}\sqrt{\beta}\sin\phi \qquad (9)$$

setting, $\phi = 0$, $\beta = \beta_0$, $\alpha = \alpha_0$ & $u(0) = u_0$ $u'(0) = u'_0$

we get $\tilde{A} = u_0/\sqrt{\beta}$ & $\tilde{B} = \sqrt{\beta_0}\, u'_0 + (\alpha_0/\sqrt{\beta_0})\, u_0$

Then
$$u(s) = \sqrt{\frac{\beta}{\beta_0}}\left(\cos\phi + \alpha_0 \sin\phi\right) u_0 + \sqrt{\beta\beta_0}\,\sin\phi\; u'_0$$

$$u'(s) = \frac{1}{\sqrt{\beta_0\beta}}\left\{(\alpha - \alpha_0)\cos\phi - (1 + \alpha\alpha_0)\sin\phi\right\} u_0$$
$$+ \sqrt{\frac{\beta_0}{\beta}}\left(\cos\phi - \alpha\sin\phi\right) u'_0$$

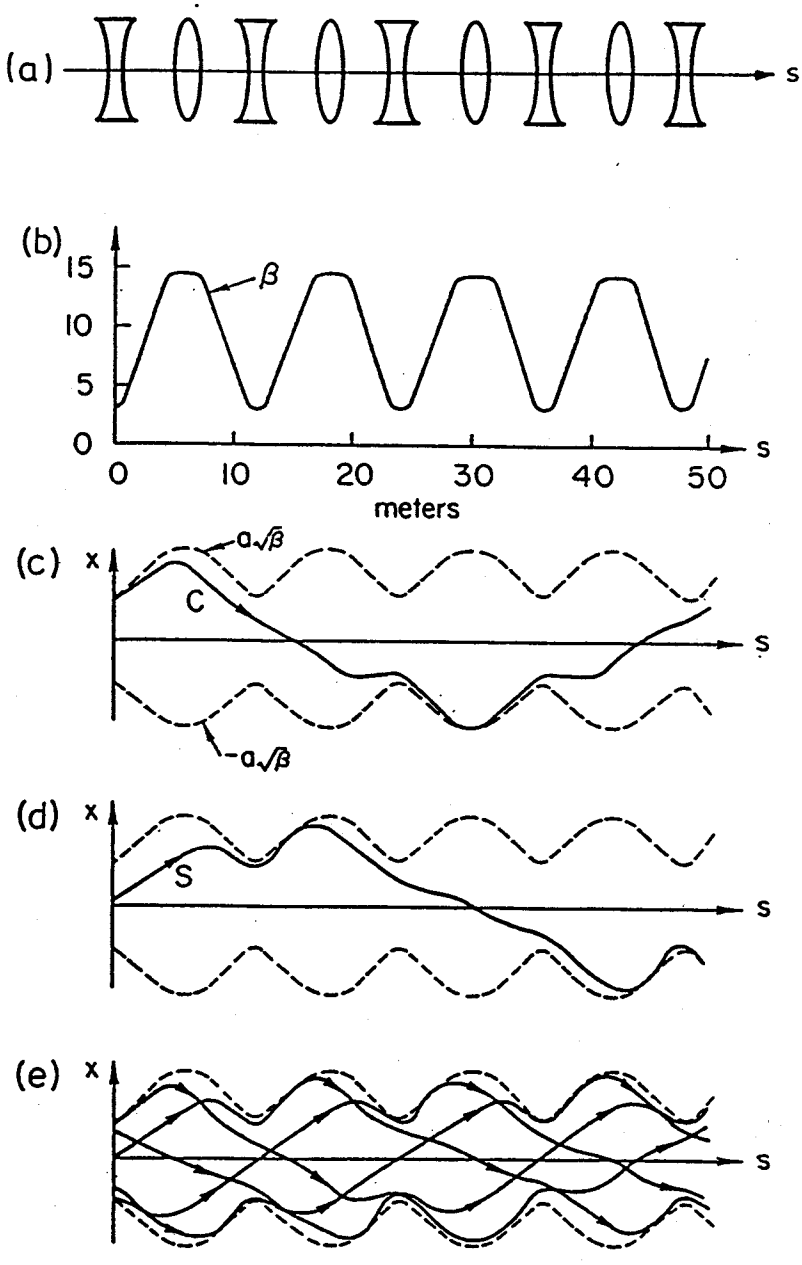


Figure 28: **a)** A regular FODO lattice of focusing and defocusing lenses **b)** Beta-function. **c)** Cosine-like trajectory for $s = 0$. **d)** Sine-like trajectory for $s = 0$. **e)** One trajectory on several successive revolutions.
(According to M. Sands, The Physic of Electron Storage Rings, SLAC–121).

Thus the transformation matrix is

$$\begin{bmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{bmatrix} = \begin{bmatrix} \sqrt{\beta/\beta_0}\,(\cos\phi + \alpha\sin\phi) & \sqrt{\beta\beta_0}\,\sin\phi \\ \frac{\alpha_0-\alpha}{\sqrt{\beta\beta_0}}\cos\phi - \frac{1+\alpha\alpha_0}{\sqrt{\beta\beta_0}}\sin\phi & \sqrt{\beta/\beta_0}\,(\cos\phi - \alpha\sin\phi) \end{bmatrix} \quad \text{—— (10)}$$

Tune of An Accelerator :-

The betatron phase for one full turn around a circular accelerator of circumference C is

$$\phi_{x,y}(C) = \int_{S}^{S+C} \frac{ds}{\beta_{x,y}(s)}$$

If this quantity is devided by 2π we get number of betatron oscillations per turn around the ring, called "tune" of the ring accelerator.

$$\nu_{x,y} = \frac{1}{2\pi}\oint \frac{ds}{\beta_{x,y}(s)} \quad \text{—— (11)}$$

one value for each transverse plane.

Envelope:- If ϵ is emittance of a beam then

$$E(s) = \sqrt{\epsilon\,\beta(s)}$$

is called envelope. This gives the maximum extent of a beam particle can go transversly.

Using eq^n (6), definitions of α, β and $E(s)$ we can get Envelope equation

$$E''(s) + E(s)K - \frac{\epsilon^2}{E^3(s)} = 0$$

Momentum Dispersion :-

Now we want to investigate the motion of particles with momenta different from a reference particle. We will find that these off-momentum particles in general undergo betatron oscillations about a new class of closed orbits in circular accelerators. The displacement of these closed orbits from that of the ideal particle will be described by a new lattice function - "dispersion function"

As outlined earlier the eq^n of motion we are interested in

$$u'' + K(s)u = \frac{\delta}{\rho} \quad ; \quad \delta = \frac{\Delta p}{p}$$

The solution of this equation can be regarded as sum of two parts

$$u(s) = \underbrace{u_\beta(s)}_{\substack{\text{solution to} \\ \text{eq}^n \text{ of motion} \\ \text{with RHS} = 0}} + \underbrace{D(s)\frac{dp}{p}}_{\substack{\text{Momentum} \\ \text{dependent} \\ \text{term only.}}} \qquad (12)$$

It can be shown that, the dispersion function

$$D(s) = \frac{\sqrt{\beta(s)}}{2\pi \sin \pi\nu} \int_s^{s+L} \frac{\beta(t)}{\rho(t)} \cos(\phi(t) - \phi(s) - \pi\nu)\, dt \qquad (13)$$

Implies that the trajectory receives additional

Kick only in the bending magnets.

Averaging the square of the expression (12) for rms displacement we can show that the total beam size due to both betatron oscillations and momentum spread as

$$\sigma_{x,y}^2 = \frac{\epsilon_{x,y}\beta_{x,y}}{\pi} + D_{x,y}^2 \left\langle \left(\frac{\Delta p}{p}\right)^2 \right\rangle \quad \text{— (14)}$$

<u>Momentum Compaction:-</u>

Consider a particle with a fixed energy which moves along a closed trajectory. Then the integral of the curvature is

$$\oint \frac{ds}{\rho} = 2\pi$$

Since $\frac{1}{\rho} = \frac{q}{p} B_\perp$

$$p = \frac{q}{2\pi} \int B_\perp \, ds$$

Leaving magnetic field unchanged consider another particle with a closed trajectory but with momentum $p + \Delta p$. This closed orbit must be different from that of particle with momentum p.

By varying p the

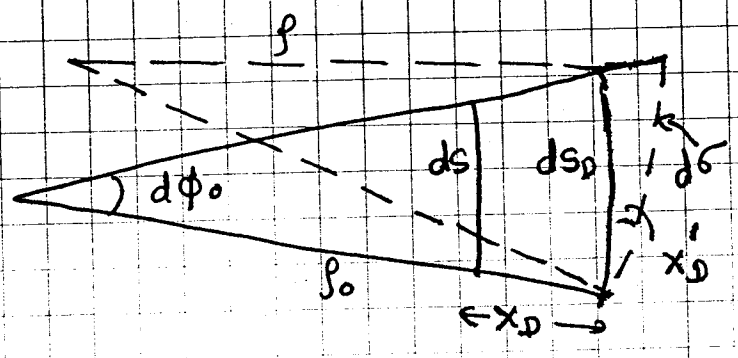
$$L = \oint ds$$

of the closed orbit can be varied. Then we can define a quantity,

$$\alpha_p = \frac{dL}{L} \Big/ \frac{dp}{p} = \frac{p}{L}\frac{dL}{dp} \qquad (15)$$

which is the ratio of fractional deviation of the path length to the corresponding fractional deviation of the momentum.

Momentum Compaction and Dispersion function :-



As we have seen earlier the differential trajectory length

$$d\sigma = \left(1 + \frac{x}{\rho}\right) ds$$

or $$L + \Delta L = \int \left(1 + \frac{x}{\rho}\right) ds$$

or $$\Delta L = \int \frac{x}{\rho}\, ds \qquad \text{with } x = D(s)\frac{dp}{p}$$

$$\frac{\Delta L}{L} = \frac{1}{L}\int \frac{D(s)}{\rho}\, ds \cdot \frac{\Delta p}{p}$$

Thus we arrive to an expression

$$\alpha = \frac{1}{L} \int \frac{D(s)\, ds}{\rho} \quad \text{— (16)}$$

Transition Energy :-

Let τ be the time interval between passage of two points separated by distance L. Then

$$\tau = \frac{L}{v} = \frac{L}{c\beta} \qquad \beta = v/c$$

Then fractional change in τ associated with deviation with L and β is

$$d\tau = \frac{dL}{c\beta} - \frac{L}{c\beta} \cdot \frac{d\beta}{\beta}$$

or $$\frac{d\tau}{\tau} = \frac{dL}{L} - \frac{d\beta}{\beta}$$

Now from eq^{n} 15 and from relativity

$$\frac{d\tau}{\tau} = \left(\alpha - \frac{1}{\gamma^2}\right) \frac{dp}{p} \qquad \gamma = \sqrt{\frac{1}{1-\beta^2}}$$

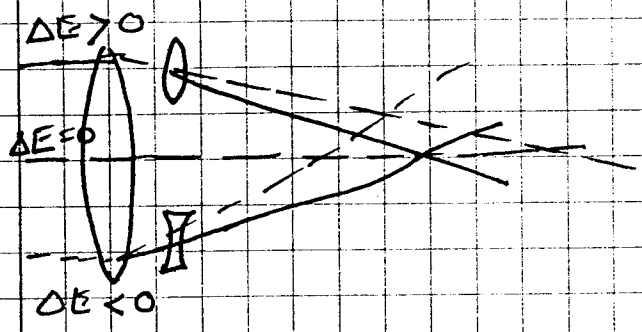
Then we write

$$\frac{d\tau}{\tau} = \left(\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}\right) \frac{dp}{p} = \eta \frac{dp}{p} \quad \text{(17)}$$

The quantity $\gamma_t = \sqrt{1/\alpha}$ is called transition energy

Chromaticity :-

Lowest order chromatic perturbation is caused by the variation of the focal length of quadrupoles with energy. In a transport line or accelerators we have focus higher energy particles more and lowenergy particles less as illustrated here.



Thus the betatron frequency depends on the momentum of the particle. The change in tune due to momentum is called Chromaticity.

$$d\nu_{x,y} = \xi_{x,y} \frac{dp}{p} \qquad (18)$$

It can be shown that

$$\xi_{x,y} = \mp \frac{1}{4\pi} \oint \underbrace{\beta_{x,y} (K}_{\text{Natural chromaticity}} - m D_{x,y})\, ds$$

which is always -ve.