Longitudinal Beam Dynamics

Particle Acceleration in a Circular Accelerator:

We have seen that an additional energy gain in an electric field V is

$$\Delta E = eV$$

for a particle with electric charge 'e'.

In modern accelerators, RF source is used to accelerate particle. Hence it is a time varying electric field; energy gain depends on the phase of the RF voltage source with respect to a particle. Thus the energy gain per turn is

$$\Delta E = eV \sin \phi_s \quad \text{—— (1)}$$

with RF frequency $\omega_{rf} = 2\pi f_{rf}$ is choosen to be an integral multiple of revolution frequency of the particle in the accelerator

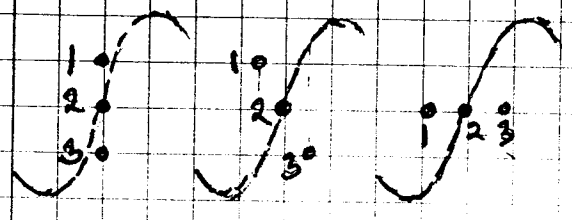
$$f_{rf} = h\, f_{rev} \quad \text{—— (2)}$$

h is called harmonic number. This number represents maximum number of bunches in the circular accelerator with f_{rf}.

Phase Stability & Phase Focusing:

In an accelerator all particles are not accelerated by the same amount. Some particles are slightly more or less accelerated relative to an ideal particle depending upon phase of arrival at the accelerating gap. A particle with right energy that gains same amount of nominal energy during acceleration process is called synchronous particle.

Let us consider a case of a particle(1) with higher momentum but crossing an accelerating gap at the same time as synchronous particle(2). This particle arrives at the next accelerating gap earlier and receives lesser energy gain relative to the synchronous particle.



As a result of this "1" will be lagging behind relative to "2". Conversely particle "3" will be ahead of "2". At some time later "3" will get less kick because it is arriving at the gap earlier and "1" will get more rf kick because it is arriving later. These particles constitute a bunch that remains synchronized with accelerating wave. This property is called phase focusing and this is phase stability.

Equation of Motion in Longitudinal Phase Space

We have seen that a synchronous particle arriving at an accelerating gap at a phase ϕ_s always receives an energy gain

$$\Delta E = e\hat{V}\sin\phi_s$$ — Subscript 's' for synchronous particle.

Now let us investigate the situations for non-synchronous particles.

Let us define variables of any other particle as

Revolution frequency : $f_r = f_{rs} + \Delta f_r$

$\omega_r = \omega_{rs} + \Delta\omega_r$

RF phase : $\phi = \phi_s + \Delta\phi = \phi_s + \psi$

Momentum : $p = p_s + \Delta p$

Energy : $E = E_s + \Delta E$

Azimuthal Angle : $\theta = \theta_s + \Delta\theta$

The azimuthal angle $\Delta\theta$ is related to ds coordinate along the path, by,

$$ds = R\,d\theta \quad \text{—} (3)$$

Over one revolution this angle "θ" changes by 2π angle, while RF phase ϕ changes by $h \times 2\pi$. Hence,

$$\Delta\phi = -h\,\Delta\theta \quad \text{—} (4)$$

The $-$ve sign comes from the fact that a particle behind the synchronous particle arrives later in time. But $\Delta t > 0$.

The angular velocity ω is given by

$$\omega = \frac{d\theta}{dt} \Rightarrow \Delta\omega = \frac{d}{dt}\Delta\theta = -\frac{1}{h}\frac{d}{dt}\Delta\phi$$

or $$\Delta\omega = -\frac{1}{h}\frac{d\psi}{dt} = -\frac{1}{h}\frac{d(\phi-\phi_s)}{dt} = -\frac{1}{h}\frac{d\phi}{dt} \quad (5)$$

The definition of η - "slip factor" was

$$\eta = \frac{1}{\gamma^2} - \frac{1}{\gamma_t^2} \quad \& \quad \frac{\Delta\tau}{\tau} = -\frac{\Delta f_r}{f_r} = -\eta\frac{\Delta p}{p}$$

$$\Delta p = \frac{p_s}{\eta\,\omega_s}\Delta\omega \qquad \omega_s = 2\pi f_s \quad (6)$$

Using (5) and (6) we have

$$\Delta p = -\frac{p_s}{h\eta\,\omega_s}\frac{d\psi}{dt} \quad (7)$$

But ΔE and Δp are related by

$$\Delta E = \beta_s c\,\Delta p = v_s\Delta p = \omega_s R_s \Delta p$$

Then eq^n 7 becomes.

$$\frac{\Delta E}{\omega_s} = -\frac{p_s R_s}{h\eta\,\omega_s}\frac{d\psi}{dt} \quad \text{— (8)}$$

As per eq^n (1), the average energy per turn is

$$(\Delta E)_{turn} = e\hat{V}\sin\phi$$

which corresponds to a momentum increment

$$(\Delta p)_{turn} = \frac{e\hat{V}}{\omega_s R}\sin\phi$$

Thus rate of change of momentum is

$$\frac{dp}{dt} = \frac{1}{\tau}\frac{e\hat{V}}{\omega_s R}\sin\phi$$

i.e $$\dot{p} = \frac{e\hat{V}}{2\pi R}\sin\phi$$

or $$2\pi(R\dot{p}) = e\hat{V}\sin\phi \quad \text{— (9)}$$

For synchronous particle it is

$$2\pi(R_s\dot{p}_s) = e\hat{V}\sin\phi_s \quad \text{— (10)}$$

Eq. (9) − Eq. (10) gives

$$2\pi\,\Delta(R\dot{p}) = e\hat{V}(\sin\phi - \sin\phi_s) \quad \text{— (11)}$$

LHS of (11) is

$$\Delta(R\dot{p}) = R\dot{p} - R_s\dot{p}_s$$

$$= (R_s + \Delta R)(\dot{p}_s + \Delta\dot{p}) - R_s\dot{p}_s$$

$$\simeq \Delta R\,\dot{p}_s + \Delta\dot{p}\,R_s$$

$$= \dot{p}_s\left(\frac{dR}{dp}\right)_s \Delta p + R_s\Delta\dot{p}$$

$$\simeq \dot{R}_s\,\Delta p + R_s\Delta\dot{p} = \frac{d}{dt}\left(R_s\Delta p\right)$$

$\therefore$

$$2\pi\,\Delta(R\dot{p}) = 2\pi\frac{d}{dt}\left(\cancel{R_s}\frac{\Delta E}{\omega_s \cancel{R_s}}\right) = 2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_s}\right)$$

Thus the motion of any arbitrary particle in terms of deviation from the synchronous particle can be expressed as

$$\left.\begin{aligned} \frac{dW}{dt} &= e\hat{V}\left(\sin\phi - \sin\phi_s\right) \\ \text{and}\quad \frac{d\psi}{dt} &= -\frac{1}{2\pi}\,\frac{h\eta\,\omega_s}{p_s R_s}\,W \\ \text{where}\quad W &= 2\pi R_s\Delta p = 2\pi\frac{\Delta E}{\omega_s} \end{aligned}\right\} \quad (12)$$

Using two equations in (12) we can get

$$\boxed{\frac{d}{dt}\left(\frac{R_s P_s}{h\eta\omega_s}\frac{d\psi}{dt}\right) + \frac{e\hat{V}}{2\pi}\left(\sin\phi - \sin\phi_s\right) = 0} \quad -(13)$$

Eq^n (12) or (13) constitute eq^n of motion in longitudinal phase space.

<u>Small Amplitude Oscillation</u> :-

consider a case with R_s, P_s, η, ω_s and $\hat{V}$ are all constant (or vary slowly w.r.t time) and $\Delta\phi = \phi - \phi_s \ll 1$ then eq^n (13) becomes

$$\frac{d^2\psi}{dt} + \frac{e\hat{V}h\eta\omega_s\cos\phi_s}{2\pi R_s P_s}\cdot\psi = 0$$

or $\ddot{\psi} + \Omega_s^2\psi = 0$ with $\Omega_s = \sqrt{\dfrac{e\hat{V}h\eta\omega_s\cos\phi_s}{2\pi R_s P_s}}$

Ω_s is synchroton oscillation frequency

In order to get a stable longitudinal motion Ω_s should be real i.e., $\eta\cos\phi_s$ must be +ve

What does $\eta \cos\phi_s > 0$ means?

since

$$\eta = \frac{1}{\gamma^2} - \frac{1}{\gamma_{tr}^2},$$

If

$$\gamma < \gamma_{tr} \qquad \eta > 0 \qquad 0 < \phi_s < \pi/2$$

$$\gamma > \gamma_{tr} \qquad \eta < 0 \qquad \pi/2 < \phi_s < \pi$$

Hence during RF acceleration we must jump the accelerating phase angle ϕ_s from $\phi_s \longrightarrow \pi - \phi_s$. This is called transition γ- phase jump.

The Hamiltonian and Phase Ellipse

We have seen that equation of motion ~~of~~ of a non-synchronous particle in longitudinal phase space is given by eq$^{\underline{n}}$ (12). These equations can be derived from, Hamiltonian defined by,

$$H(\phi, W, t) = -\frac{1}{4\pi}\frac{h\eta\omega_s}{p_s R_s} W^2 + e\hat{V}\left[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s\right] \quad \text{— (16)}$$

This follows from Hamiltonian equation of motion

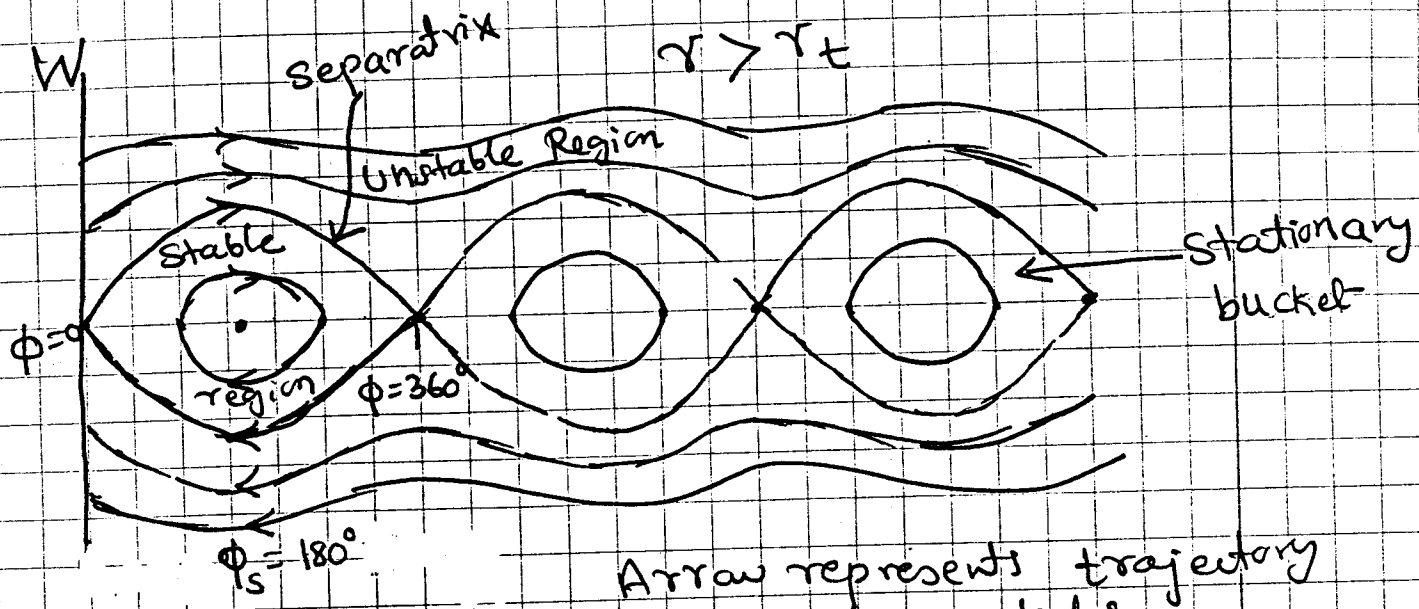
$$\frac{\partial H}{\partial \phi} = -\frac{\partial W}{\partial t} = eV\{\sin\phi + \sin\phi_s\}$$

$$\frac{\partial H}{\partial W} = \frac{\partial \phi}{\partial t} = -\frac{h\eta\omega_s W}{2\pi p_s R_s}$$

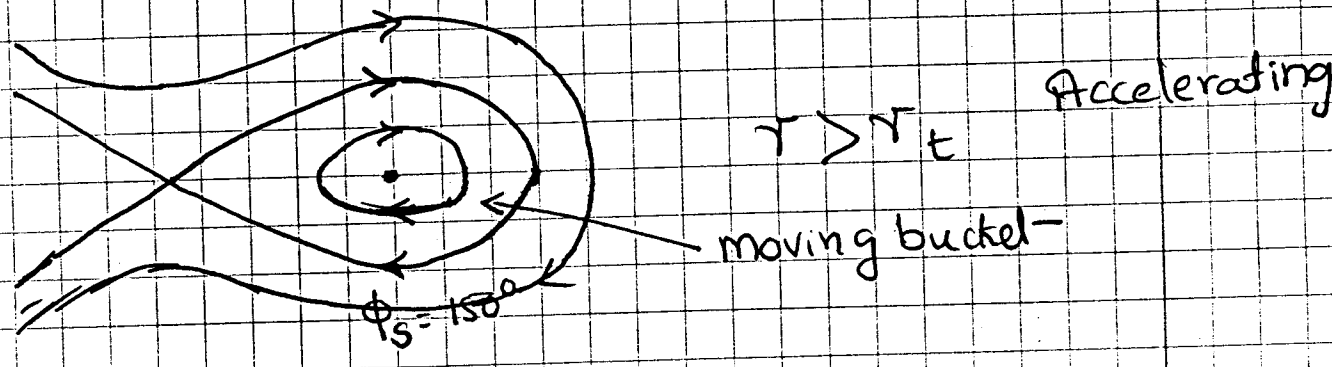
Then the motion of the particles are conveniently represented by their trajectories in (W, ϕ) space. The trajectories are contours of constant Hamiltonian given by eq$^{\underline{n}}$ (16). Thus

$$H \equiv -AW^2 + B\left[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s\right]$$

$$W = \pm\sqrt{\frac{B}{A}\left[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s\right] - \frac{H}{A}} \quad \text{— (17)}$$



Arrow represents trajectory of particles.



Stationary Buckets & Beam in the buckets

Bucket Area $S = 8\beta_s \sqrt{\dfrac{2E_s V}{\pi h \eta}}$ eV-sec

Bucket Height $= \pm \beta_s \sqrt{\dfrac{2E_s V}{\pi h \eta}}$ eV-sec

Beam Area $\varepsilon_l = S \times \dfrac{\pi}{64} \Delta^2 \left(1 - \dfrac{5}{384}\Delta^2\right)$; $\Delta \lesssim 4$ radia

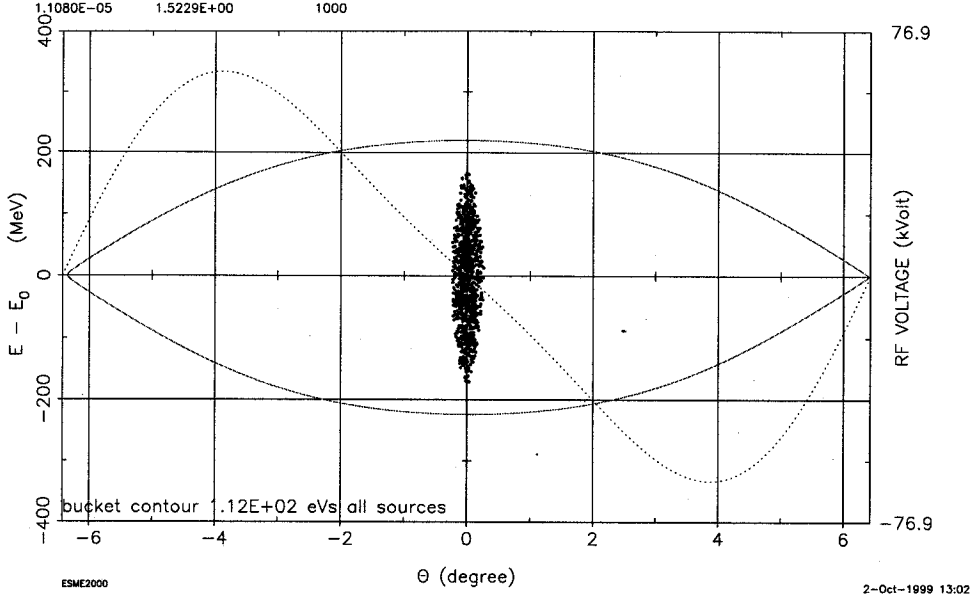
$$\frac{\Delta p}{p} = \left(\frac{2\beta_s^2 E_s eV}{\pi h \eta}\right)^{1/2} \frac{1}{\beta_s^2 E_s} \sin\frac{\delta\phi}{4} \; ; \; \delta\phi = 2\pi f_{rf}(F.W)$$

F.W = Δt of the bunch

MI deceleration: Bunch Rotation at 25 GeV with h=

TURN 294666 3.263E+00 sec

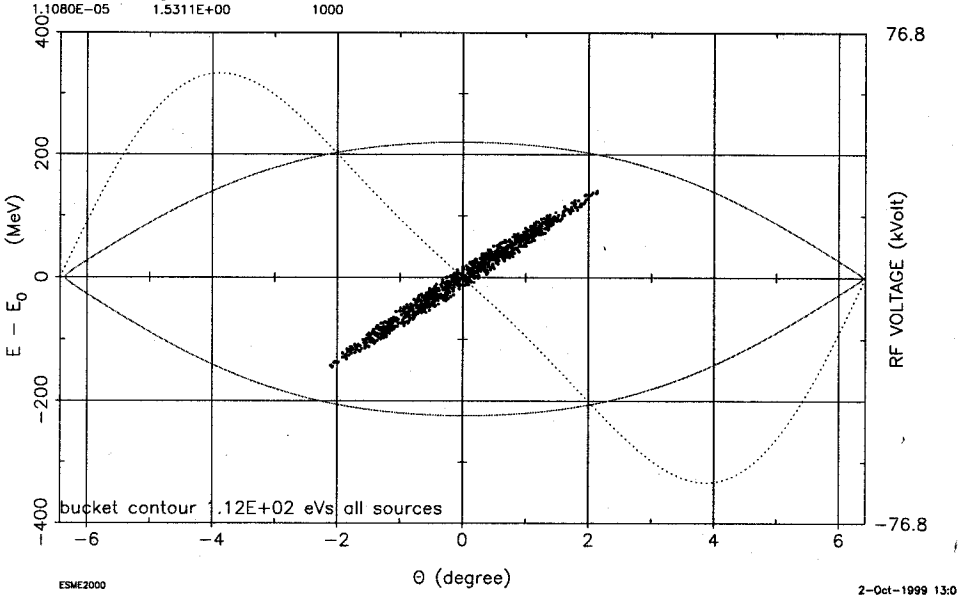
H_B (MeV)	S_B (eV s)	E_S (MeV)	h	V (MV)	ψ (deg)
2.2208E+02	1.1184E+02	2.5018E+04	28	6.000E−02	1.800E+02
ν_S (turn^{-1})	pdot (MeV s^{-1})	η	56	1.200E−02	3.025E−02
8.5980E−05	−8.5842E−01	6.9073E−04			
τ (s)	S_b (eV s)	N			
1.1080E−05	1.5229E+00	1000			



MI deceleration: Bunch Rotation at 25 GeV with h=

TURN 296000 3.278E+00 sec

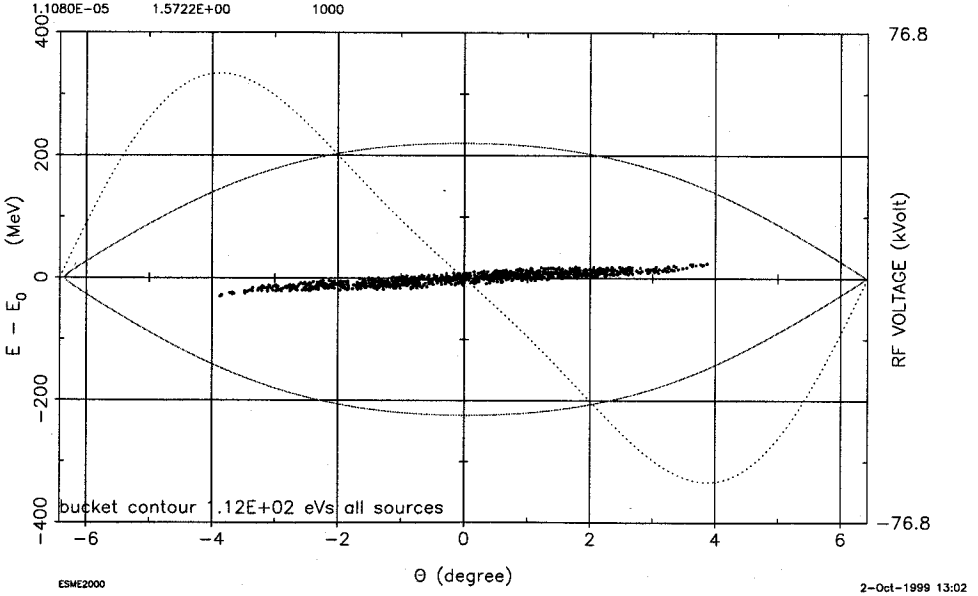
H_B (MeV)	S_B (eV s)	E_S (MeV)	h	V (MV)	ψ (deg)
2.2192E+02	1.1175E+02	2.5018E+04	28	5.991E−02	1.800E+02
ν_S (turn^{-1})	pdot (MeV s^{-1})	η	56	1.198E−02	3.030E−02
8.5917E−05	−8.5834E−01	6.9073E−04			
τ (s)	S_b (eV s)	N			
1.1080E−05	1.5311E+00	1000			



MI deceleration: Bunch Rotation at 25 GeV with h=

TURN 298000 3.300E+00 sec

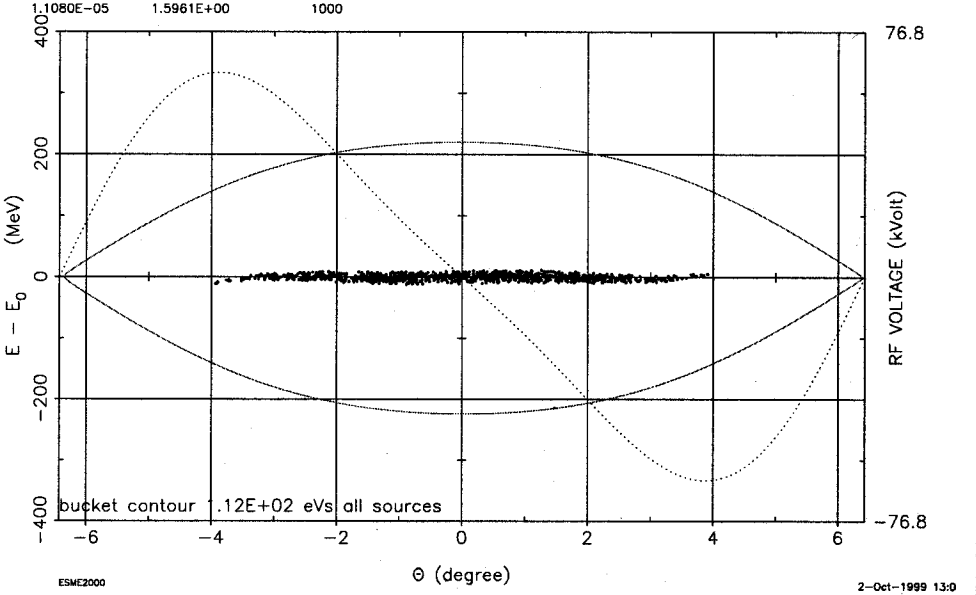
H_B (MeV)	S_B (eV s)	E_S (MeV)	h	V (MV)	ψ (deg)
2.2192E+02	1.1175E+02	2.5018E+04	28	5.991E−02	1.800E+02
ν_S (turn^{-1})	pdot (MeV s^{-1})	η	56	1.198E−02	3.030E−02
8.5917E−05	−8.5834E−01	6.9073E−04			
τ (s)	S_b (eV s)	N			
1.1080E−05	1.5722E+00	1000			



MI deceleration: Bunch Rotation at 25 GeV with h=

TURN 298294 3.303E+00 sec

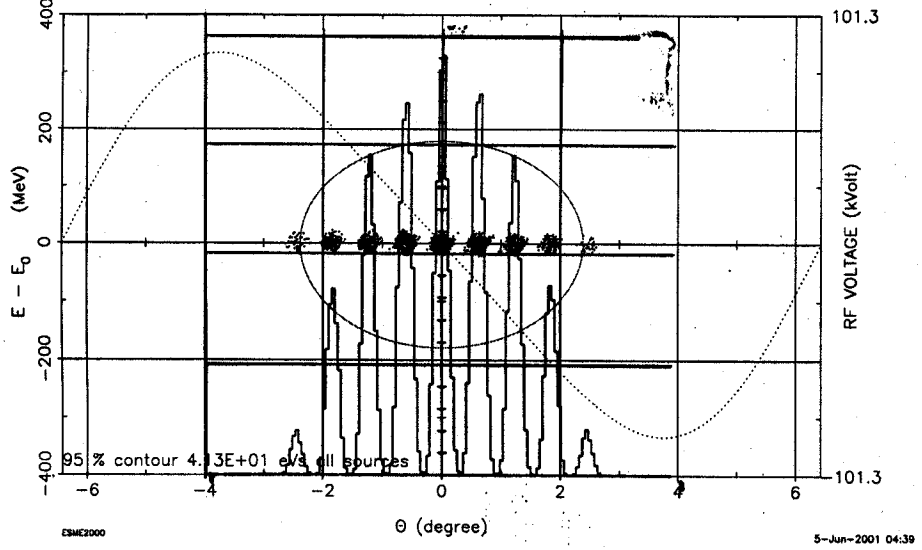
H_B (MeV)	S_B (eV s)	E_S (MeV)	h	V (MV)	ψ (deg)
2.2192E+02	1.1175E+02	2.5018E+04	28	5.991E−02	1.800E+02
ν_S (turn^{-1})	pdot (MeV s^{-1})	η	56	1.198E−02	3.030E−02
8.5917E−05	−8.5834E−01	6.9073E−04			
τ (s)	S_b (eV s)	N			
1.1080E−05	1.5961E+00	1000			



Coalescing in 2.5 MHz Buckets

TURN 412000 5.564E+00 sec

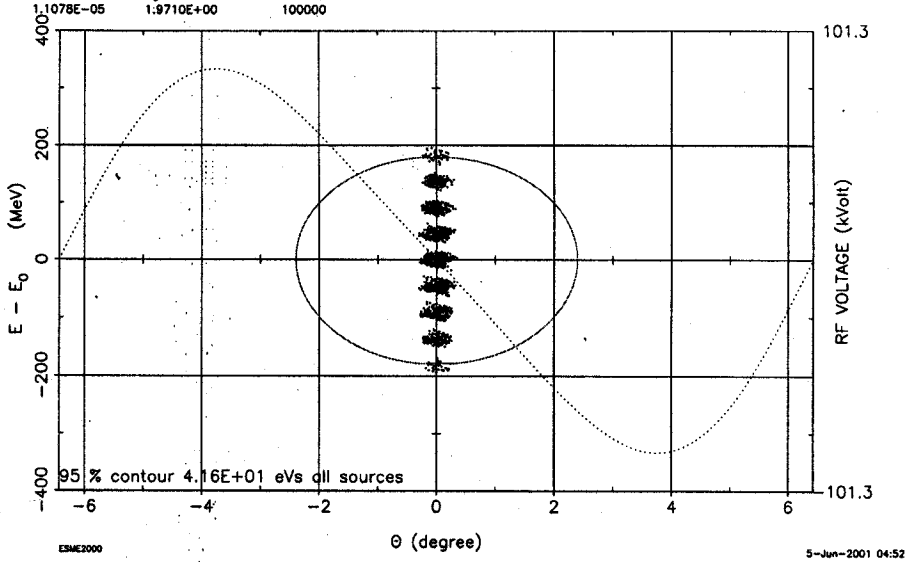
H_B (MeV)	S_B (eV s)	E_S (MeV)	h	V (MV)	ψ (deg)
3.6636E+02	1.8456E+02	1.5000E+05	28	8.100E-02	1.800E+02
ν_S (turn^{-1})	pdot (MeV s^{-1})	η	56	1.228E-02	0.000E+00
7.0377E-05	8.9544E-13	2.0581E-03			
τ (s)	S_b (eV s)	N			
1.1078E-05	1.8523E+00	100000			



End of Rotation in 2.5 MHz Buckets

TURN 416202 5.611E+00 sec

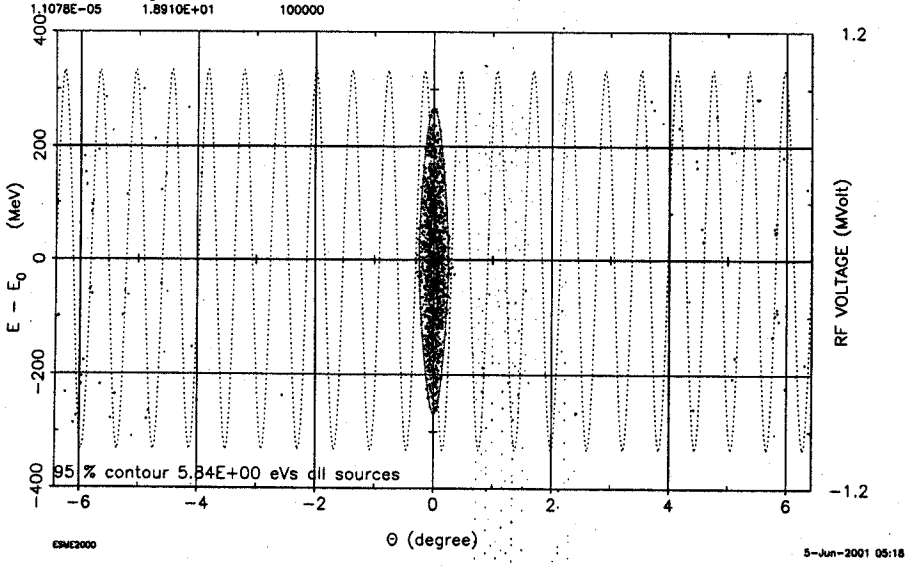
H_B (MeV)	S_B (eV s)	E_S (MeV)	h	V (MV)	ψ (deg)
3.6581E+02	1.9344E+02	1.5000E+05	28	8.100E-02	1.800E+02
ν_S (turn^{-1})	pdot (MeV s^{-1})	η	56	1.228E-02	0.000E+00
7.0377E-05	8.9544E-13	2.0581E-03			
τ (s)	S_b (eV s)	N			
1.1078E-05	1.9710E+00	100000			



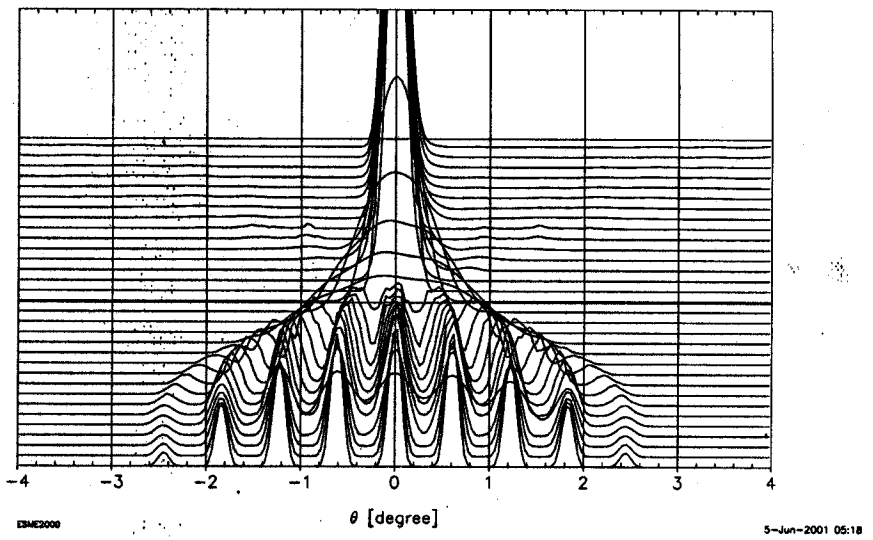
End of Rotation in 2.5 MHz Buckets

TURN 424245 5.700E+00 sec

H_B (MeV)	S_B (eV s)	E_S (MeV)	h	V (MV)	ψ (deg)
2.8090E+02	6.7384E+00	1.5000E+05	588	1.000E+00	1.800E+02
ν_S (turn^{-1})	pdot (MeV s^{-1})	η	1176	0.000E+00	0.000E+00
1.1332E-03	1.1055E-11	2.0581E-03			
τ (s)	S_b (eV s)	N			
1.1078E-05	1.8910E+01	100000			



End of Rotation in 2.5 MHz Buckets

every 330 turns, from turn 410520

Beam Current Profiles